Moment Inequalities of Overpartition Cranks

Amherst REU Seminar

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Definition

A *partition* π is a non-increasing sequence of positive integers. If the sum of these integers is n, then we write $\pi \vdash n$, or $|\pi| = n$. Let $p(n)$ denote the number of partitions of n.

The partitions $\pi \vdash 4$ are

$$\begin{array}{ll} (4) & (3,1) \\ (2,2) & (2,1,1) \\ (1,1,1,1). & \end{array}$$

Thus, $p(4) = 5$.

Because $p(n)$ is finite for all n, we may index summations over the set of partitions. Define

$$P(q) = \sum_{\pi} q^{|\pi|} = \sum_{n=0}^{\infty} p(n) q^n.$$

This is the *generating series* for $p(n)$. Euler proved that

$$P(q) = \prod_{i=1}^{\infty} \frac{1}{1-q^i}.$$

Proof. Multiplication of generating series is analogous to taking a Cartesian product.

$$\prod_{i=1}^{\infty}\frac{1}{1-q^i}=\prod_{i=1}^{\infty}(1+q^i+q^{2i}+q^{3i}+\cdots)$$
$$=\prod_{i=1}^{\infty}(1+q^i+q^{i+i}+q^{i+i+i}+\cdots)=\sum_{\pi}q^{|\pi|}. \quad \square$$

We will abbreviate these kinds of products using the q-Pochhamer symbol

$$(a;q)_n=\prod_{i=0}^{n-1}(1-aq^i).$$

Thus $P(q)=1/(q;q)_\infty$.

Theorem (Ramanujan, Hardy, 1920)

For all $n \geq 0$,

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod 5 \\ p(7n+5) &\equiv 0 \pmod 7 \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

Freeman Dyson developed the *rank* function to give a combinatorial proof of the Ramanujan congruences. The rank of a partition π is defined to be the largest part of π minus the number of parts of π. For example,

$$r((4,3,2)) = 4 - 3 = 1.$$

Divvying the partitions $\pi \vdash (5n+4)$ according to their rank modulo 5 produces five sets of equal size. This technique also proves the modulo 7 congruence, but it fails for the modulo 11 congruence. What fills the gap?

Frank Garvan and George Andrews studied the two-variable series

$$C(z;q) := \frac{(q;q)_\infty}{(zq, q/z; q)_\infty} = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} M(m,n) z^m q^n$$

looking for a solution. Note that $C(1;q) = P(q)$, implying that

$$\sum_m M(m,n) = p(n).$$

A combinatorial interpretation of $M(m,n)$ as the number of π with some statistic equal to m could hold the answer.

Crank Function

Definition

If a partition π does not contain any 1s, then the *crank* of π is defined to be the largest part of π.
Otherwise, let $w(\pi)$ denote the number of 1's occurring in π, and let $\mu(\pi)$ denote the number of parts of π which are larger than $w(\pi)$. In this case, the crank of π is defined to be

$$c(\pi) = \mu(\pi) - w(\pi).$$

For example, if $\pi = (3, 2, 1, 1)$, then $w(\pi) = 2$, $\mu(\pi) = 1$, and $c(\pi) = -1$. An easier example is

$$c((4, 3, 2)) = 4.$$

Andrews and Garvan proved that $M(m, n)$ is the number of partitions $\pi \vdash n$ with $c(\pi) = m$. Not only does this give a proof of the modulo 11 Ramanujan congruence, it gives a proof of all the Ramanujan congruences.

The partitions of 4, their ranks, and cranks are given below.

π	$r(\pi)$	$c(\pi)$
(4)	3	4
$(2,2)$	0	2
$(3,1)$	1	0
$(2,1,1)$	-1	-2
$(1,1,1,1)$	-3	-4

The ℓth *moment* of the partition crank is defined to be

$$M_\ell(n) := \sum_{\pi \vdash n} c(\pi)^\ell.$$

Theorem (Dyson, 1989)

For $n \geq 0$,

$$M_2(n) = 2np(n).$$

Overpartitions

Definition

A *overpartition* is a non-increasing sequence of positive integers, where the first occurrence of each part may be overlined.

The overpartitions $\pi \vdash 3$ are

$$\begin{array}{llll} (3) & (\overline{3}) & (1,1,1) & (\overline{1},1,1) \\ (2,1) & (2,\overline{1}) & (\overline{2},1) & (\overline{2},\overline{1}). \end{array}$$

The generating series for overpartitions is

$$\overline{P}(q) = \sum_{\pi} q^{|\pi|} = \frac{(-q;q)_\infty}{(q;q)_\infty} = \sum_{n=0}^{\infty} \overline{p}(n) q^n.$$

Residual Cranks

Definition

For $k \geq 1$, the kth residual partition of π is a partition π' consisting of $1/k$th of each of the non-overlined parts of π that are divisible by k. The kth *residual crank* of π is then defined to be $c_k(\pi) = c(\pi')$.

For example,

$$\begin{aligned}
c_1((4, \overline{3}, 2)) &= c((4, 2)) = 4 \\
c_2((4, \overline{3}, 2)) &= c((2, 1)) = 0 \\
c_3((4, \overline{3}, 2)) &= c(\emptyset) = 0 \\
c_4((4, \overline{3}, 2)) &= c((1)) = -1 \\
c_5((4, \overline{3}, 2)) &= c(\emptyset) = 0.
\end{aligned}$$

Let $\overline{M[k]}_\ell(n)$ denote the ℓth moment of the kth residual crank function.

Theorem (Larson, Rust, Swisher, 2014)

For $n \geq 2$,

$$\overline{M[2]}_1^+(n) < \overline{M[1]}_1^+(n).$$

Theorem (Al-Saedi, Swisher, M., 2020)

For $k, \ell \geq 1$,

$$\overline{M[k+1]}_\ell^+(n) \leq \overline{M[k]}_\ell^+(n),$$

with equality if and only if $n < k$, in which case $\overline{M[k]}_\ell^+(n) = 0$.

Definition

Let $nov_k(n)$ denote the sum of all non-overlined parts which vanish modulo k, taken across all overpartitions $\pi \vdash n$.

We see that $nov_2(3) = 6$

$$\begin{array}{llll} (3) & (\overline{3}) & (1,1,1) & (\overline{1},1,1) \\ (2,1) & (2,\overline{1}) & (\overline{2},1) & (\overline{2},\overline{1}). \end{array}$$

Theorem (M., Simonič, 2021)

For $n \geq 0$ *and* $k \geq 1$,

$$k \cdot \overline{M[k]}_2(n) = 2 \cdot nov_k(n).$$

Proof. The generating series for the partition crank function is

$$\frac{(q;q)_\infty}{(zq,q/z;q)_\infty}.$$

Any overpartition π' may be divvied into the parts that contribute to its kth residual partition, and those that do not.

Parts of π which contribute to π' are generated by

$$\frac{(q^k;q^k)_\infty}{(zq^k,q^k/z;q^k)_\infty}.$$

Parts that do not are generated by

$$(-q;q)_\infty \frac{(q^k;q^k)_\infty}{(q;q)_\infty}.$$

Thus the generating series for the kth residual crank is

$$\frac{(-q;q)_\infty}{(q;q)_\infty} \frac{(q^k,q^k;q^k)_\infty}{(zq^k,q^k/z;q^k)_\infty}.$$

We can calculate the generating series of the second crank moment by using the operator $z\frac{\partial}{\partial z}$, since

$$\left(z\frac{\partial}{\partial z}\right)^2\left\{\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} M(m,n)z^m q^n\right\} = \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} m^2 M(m,n)z^m q^n.$$

Substituting $z = 1$ completes the calculation.

For our series,

$$\left(z\frac{\partial}{\partial z}\right)^2 \frac{(-q;q)_\infty}{(q;q)_\infty} \frac{(q^k,q^k;q^k)_\infty}{(zq^k,q^k/z;q^k)_\infty}$$
$$= (-q;q)_\infty \frac{(q^k;q^k)_\infty}{(q;q)_\infty} \left(z\frac{\partial}{\partial z}\right)^2 \frac{(q^k;q^k)_\infty}{(zq^k,q^k/z;q^k)_\infty}$$
$$= (-q;q)_\infty \frac{(q^k;q^k)_\infty}{(q;q)_\infty} \left(z\frac{\partial}{\partial z}\right)^2 C(z;q^k).$$

Substituting $z = 1$ gives us a generating series for the second moment of the kth residual crank moment in terms of the second crank moment for partitions. Multiply both sides by k.

$$\begin{aligned}\sum_{n=0}^{\infty} k \cdot \overline{M[k]}_2(n) q^n &= (-q;q)_\infty \frac{(q^k;q^k)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} k \cdot M_2(n) q^{kn} \\ &= (-q;q)_\infty \frac{(q^k;q^k)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} 2kn \cdot p(n) q^{kn} \\ &= (-q;q)_\infty \frac{(q^k;q^k)_\infty}{(q;q)_\infty} \sum_{\pi'} 2k|\pi'| q^{k|\pi'|} \\ &= \sum_{n=0}^{\infty} 2 \cdot nov_k(n) q^n. \quad \square\end{aligned}$$

Definition

Let $ov_k(n)$ denote the sum of all overlined parts which vanish modulo k, taken across all overpartitions $\pi \vdash n$.

We see that $ov_2(3) = 2$

$$\begin{array}{llll} (3) & (\overline{3}) & (1,1,1) & (\overline{1},1,1) \\ (2,1) & (2,\overline{1}) & (\overline{2},1) & (\overline{2},\overline{1}). \end{array}$$

Lemma (M., Simonič, 2021)

For $n \geq 0$ *and* $k \geq 1$,

$$ov_k(n) = nov_k(n) - nov_{2k}(n).$$

Corollary (M., Simonič, 2021)

For $n \geq 0$ *and* $d, k \geq 1$,

$$d \cdot \overline{M[dk]}_2(n) \leq \overline{M[k]}_2(n),$$

with equality if and only if $n < k$*, in which case* $\overline{M[k]}_2(n) = 0$.

Thank you!

Encore

Let $q = \exp(2\pi i z)$. For $k \geq 1$, the *Eisenstein series* $E_{2k}(z)$ may be defined as

$$E_{2k}(z) = 1 + \frac{(2\pi)^{2k}}{(-1)^k (2k-1)! \zeta(2k)} \Phi_{2k-1}(q),$$

where ζ is the Riemann zeta function and

$$\Phi_\ell(q) = \sum_{n=1}^{\infty} \left\{ \sum_{d|n} d^\ell \right\} q^n$$

is a divisor sum generating function. These are meromorphic functions for $\Im z > 0$.

For $k \geq 2$, the function E_{2k} is a *modular form*. In fact, all modular forms are generated as products of Eisenstein series. However, E_2 is not modular.

Functions generated by products of modular forms and E_2 are called *quasimodular forms*.

Let $\overline{C[k]}_\ell(q)$ denote the ℓth moment generating function for the kth residual crank.

Theorem (M., Simonič, 2021)

For $j, k \geq 1$ and $m \geq 0$, the function

$$\left(q\frac{\partial}{\partial q}\right)^m \overline{C[k]}_{2j}(q)$$

is in the space $\frac{(-q;q)_\infty}{(q;q)_\infty} \cdot \mathcal{W}_{j+m}(\Gamma_0(lcm(2,k)))$.

That is, $\left(q\frac{\partial}{\partial q}\right)^m \overline{C[k]}_{2j}(q)$ is $\overline{P}(q)$ times a quasimodular form.