Number Theory Down Under 8

Further Study of the Knave Map

Thomas Morrill[1] Mansur Boase

[1]Trine University
Slides available at `tsmorrill.github.io`.

2020 October 8

Outline

- Look-and-Say Sequences
- The Look-Knave Sequence
- Limits of $k^n(s)$.
- Asymptotics
- Preliminary Bounds
- Next Steps

Look-and-Say Sequences

1
11
21
1211
111221

Look-and-Say Sequences

The *Look-and-Say Sequence* is a sequence of integers (words)

$$\mathtt{1}, \mathtt{11}, \mathtt{21}, \mathtt{1211}, \ldots$$

in which the digits of s_n are the description of the digits of s_{n-1}.

Look-and-Say Sequences

Theorem (Conway, 1987)

Let s_n *denote the nth term in the Look-and-Say sequence. Then,*

$$\lim_{n\to\infty} \frac{|s_{n+1}|}{|s_n|} = 1.3035\ldots,$$

which is an algebraic integer of degree 71*.*

Look-and-Say Sequences

Theorem (Conway's Cosmological Theorem, 1987)

Let ℓ denote the Look-and-Say map. There is a known set (table) of words (elements) so that the following hold.

For all pairs of elements u, v on the table, $\ell(uv) = \ell(u)\ell(v)$.

Further, let $w \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}^n$, with $n \geq 1$. Then there exists $m \geq 0$ so that $\ell^m(w)$, and all its iterates under ℓ decompose into words from the table of elements

$$w' = w'_1 \dots w'_k.$$

"All compounds eventually decompose into lower elements."

Look-and-Say Sequences

The *Binary Look-and-Say Sequence* is a sequence of binary words

$$\texttt{1}, \texttt{11}, \texttt{101}, \texttt{111011}, \ldots$$

in which the bits of s_n are the base-2 description of the digits of s_{n-1}.

Look-and-Say Sequences

Theorem (Johnston, 2010 (blog post))

Let s_n denote the nth term in the Binary Look-and-Say sequence. Then,

$$\lim_{n\to\infty} \frac{|s_{n+1}|}{|s_n|} = 1.4655\ldots,$$

which is an algebraic integer of degree 3.

The Look-Knave Map

For a binary word s, let $\overline{s}$ be the bitwise complement of s.

The *Look-Knave Sequence* is a sequence of binary words, in which $s_1 = 1$, and the bits of s_n are the base-2 description of $\overline{s_{n-1}}$.

We call this operation the *knave map*,

$$k : \bigcup_{n=1}^{\infty} \{\mathtt{0},\ \mathtt{1}\}^n \to \bigcup_{n=1}^{\infty} \{\mathtt{0},\ \mathtt{1}\}^n.$$

The Look-Knave Map

Here is the calculation of the first few terms of the sequence.

$$s_1 = \mathtt{1} \qquad \overline{s_1} = \mathtt{0}$$

$$s_2 = \mathtt{10} \qquad \overline{s_2} = \mathtt{01}$$

$$s_3 = \mathtt{1011} \qquad \overline{s_3} = \mathtt{0100}$$

$$s_4 = \mathtt{1011101} \qquad \overline{s_3} = \mathtt{0100010}$$

$$\vdots \qquad \vdots$$

The Look-Knave Map

Entries of the Look-Knave sequence

1
10
1011
1011100
1011110101
1011100011101110
10111101111101111011
1011100011101011100011100
1011110111110111011110111110101
101110001110101111011100011101011101110

The Look-Knave Map

Conjecture

Let s_n denote the nth term in the Look-Knave sequence. Then,

$$\lim_{n\to\infty} \frac{|s_{n+1}|}{|s_n|} = 1.12\ldots,$$

which is an algebraic integer.

Limits of $k^n(s)$

Odd entries of the Look-Knave Sequence

1
1011
1011110101
1011110111101111011
101111011110111011110111110101

Even entries of the Look-Knave Sequence

10
1011100
1011100011101110
1011100011101011100011100
1011100011101011101110001110101110111 0

Limits of $k^n(s)$

Theorem (M, 2020)

Let $s_n = k^n(1)$. Then the bitwise limits

$$\lim_{n\to\infty} s_{2n} = S_0, \qquad \lim_{n\to\infty} s_{2n+1} = S_1$$

exist. Further, S_0 is the description of $\overline{S_1}$ and vice versa.

$$S_0 = \mathtt{101110001101}\ldots$$
$$S_1 = \mathtt{101111011110}\ldots$$

Limits of $k^n(s)$

Can we really define the action of k on any infinite binary word? Simple, we only have two cases to handle: a tail of all `0`s, and a tail of all `1`s.

$$k(\mathtt{0}\ldots) := \mathtt{1}\ldots$$
$$k(\mathtt{1}\ldots) := \mathtt{0}\ldots$$

This plays nicely with the inclusion map

$$\bigcup_{n=1}^{\infty}\{\mathtt{0},\ \mathtt{1}\}^n \to \{\mathtt{0},\ \mathtt{1}\}^{\mathbb{N}}$$
$$w\mathtt{0} \mapsto w\mathtt{0}\ \mathtt{1}\ldots$$
$$w\mathtt{1} \mapsto w\mathtt{1}\ \mathtt{0}\ldots$$

Limits of $k^n(s)$

Theorem (M, 2020)

Let S be any infinite binary word. If S is neither of $(0\ldots)$ and $(1\ldots)$, then the bitwise limits

$$\lim_{n\to\infty} k^{2n}(S), \qquad \lim_{n\to\infty} k^{2n+1}(S)$$

both exist, and

$$\left\{\lim_{n\to\infty} k^{2n}(s),\ \lim_{n\to\infty} k^{2n+1}(s)\right\} = \{S_0, S_1\}$$

Thus, S_0 and S_1 are the attracting fixed points of the map

$$k^2 : \{\texttt{0},\ \texttt{1}\}^{\mathbb{N}} \to \{\texttt{0},\ \texttt{1}\}^{\mathbb{N}},$$

and $(0\ldots)$ and $(1\ldots)$ are the repelling fixed points.

Asymptotics

What are the properties of k as a function of infinite sequences?

What are the properties of S_0 and S_1?

How about the density of 1s in these words?

Asymptotics

For a binary word s, define

$$
\begin{aligned}
|s| &= \text{the length of } s. \\
|s|_{\mathtt{1}} &= \text{the number of } \mathtt{1}\text{s which occur in } s. \\
d_{\mathtt{1}}(s) &= |s|_{\mathtt{1}}/|s|. \\
\Delta(s) &= \text{the number of bit changes which occur in } s.
\end{aligned}
$$

For example,

$$
\begin{aligned}
d_1(\mathtt{11011}) &= \frac{|\mathtt{11011}|_{\mathtt{1}}}{|\mathtt{11011}|} = \frac{4}{5} \\
\Delta(\mathtt{11011}) &= 2.
\end{aligned}
$$

Asymptotics

We are interested in studying limits of the form

$$\lim_{n\to\infty} \frac{f(w_n)}{g(w_n)},$$

where w_n is the prefix of S_0 (resp., S_1) of length n.

Notice w_n is also a prefix of some word $k^m(\mathtt{1})$!

Preliminary Bounds

Lemma

For $s_n \in \{k^n(1)\}_{n=0}^{\infty}$,

$$|s_n| \asymp |s_n|_1 \asymp |s_n|_0 \asymp \Delta(s_n) \asymp |k(s)|.$$

That is, for each pair of functions, there exist constants A, B *so that*

$$A|s_n| \leq \Delta(s_n) \leq B|s_n|$$

and so on.

Proof.

Run-lengths of s are bounded; at most three `0`s, or five `1`s, may occur in a run. □

Preliminary Bounds

Corollary

For $s_n \in \{k^n(1)\}_{n=0}^{\infty}$,

$$1 \asymp d_1(s_n) \asymp d_0(s_n) \asymp \frac{\Delta(s_n)}{|s_n|} \asymp \frac{|k(s_n)|}{|s_n|}.$$

Preliminary Bounds

Lemma (M, 2020)

All $s_n \in \{k^n(1)\}_{n=0}^{\infty}$, may be decomposed into a concatenation of subwords w listed below.

Body		Tail	
w	$k(w)$	w	$k(w)$
0	11	00	101
000	111		
1	10	11	100
111	110		
1111	1000		
11111	1010		

Here, subwords in the "tail" column may only occur as the final subword of s_n.

This means `0110` and `1001` never occur in S_0 or S_1.

Preliminary Bounds

Subwords of S_0, S_1, their densities, and densities are given below.

s	$d(s)$	$k(s)$	$d(k(s))$
`10`	$1/2$	`1011`	$3/4$
`1000`	$1/4$	`10111`	$4/5$
`1110`	$3/4$	`1011`	$4/5$
`111000`	$1/2$	`110111`	$5/6$
`11110`	$4/5$	`100011`	$1/2$
`1111000`	$4/7$	`1000111`	$4/7$
`111110`	$5/6$	`101011`	$2/3$
~~`11111000`~~			

With a little work, one can show `11111000` also does not occur in S_0 or S_1.

Preliminary Bounds

They warned me about averaging averages...

Lemma (The Mediant Inequality)

For $1 \leq i \leq n$*, let* $q_i = \frac{a_i}{b_i}$ *with* $a_i \geq 0$*,* $b_i > 0$*, and*

$$q_1 \leq ... \leq q_n.$$

Further let ω_i *be nonnegative weights. Then*

$$q_1 \leq \frac{\omega_1 a_1 + \cdots + \omega_n a_n}{\omega_1 b_1 + \cdots + \omega_n b_n} \leq q_n$$

Preliminary Bounds

So, if s is a prefix of S_0 or S_1, then

$$\frac{|s|_1}{|s|} = \frac{\omega_1 \Sigma w_1 + \cdots + \omega_n \Sigma w_n}{\omega_1 |w_1| + \cdots + \omega_n |w_n|},$$

where the w_i are subwords taken from the $k(s)$ column of the table. This gives us

$$\frac{1}{2} \leq \frac{|s|_{\mathtt{1}}}{|s|} \leq \frac{5}{6}.$$

So, there are no fewer `1`s than `0`s.

Can we do better?

Gerrymandering

Preliminary Bounds

We don't like the subword $s = \texttt{11110}$, because the density of $k(s)$ is not rigged in our favor.

s	$k(s)$	$d(k(s))$
10	1011	3/4
1000	10111	4/5
1110	1011	4/5
111000	110111	5/6
11110	100011	1/2
1111000	1000111	4/7
111110	101011	2/3

By studying the $k(s)$ column, we deduce $\texttt{11110}$ cannot immediately follow itself in $k(s_n)$, but $s_n = k(s_{n-1})$.

Preliminary Bounds

Let's replace the `11110` line in the table by considering all possible subwords which begin with `1110` instead.

s	$k(s)$	$d(k(s))$
`1111010`	`1000111011`	$6/10$
`111101000`	`10001110111`	$7/11$
`111101110`	`1000111011`	$7/11$
`11110111000`	`100011110111`	$2/3$
`111101111000`	`1000111000111`	$7/13$
`11110111110`	`100011101011`	$7/12$

This implies

$$\frac{7}{13} \leq \frac{|s|_1}{|s|} \leq \frac{5}{6}.$$

Preliminary Bounds

Let s_n be the prefix of length n of S_0 (resp., S_1). Then

$$1 \leq \underline{\lim} \frac{k^{n+1}(\mathtt{1})}{k^n(\mathtt{1})} \leq \overline{\lim} \frac{k^{n+1}(\mathtt{1})}{k^n(\mathtt{1})} \leq 1.7$$

$$\frac{7}{13} \leq \underline{\lim} \frac{|s|_{\mathtt{1}}}{|s|} \leq \overline{\lim} \frac{|s|_{\mathtt{1}}}{|s|} \leq \frac{5}{6}$$

Next Steps

- Study k^4 instead of k^2.
- Periodicity/entropy of S_0, S_1?
- Work in an extended alphabet $\{\mathit{0}, \mathit{1}, \mathbf{0}, \mathbf{1}\}$.

Thank you!