Oregon State University REU 2020

Look, Knave

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Outline

- Look-and-Say Sequences
- The Look-Knave Sequence
- Limiting behavior of the Look-Knave Sequence
- Uniqueness of S_{even} and S_{odd}

1

11

21

1211

111221

The Look-and-Say Sequence is a sequence of decimal strings

$$1, 11, 21, 1211, 111211, 311221, \ldots,$$

in which the digits of s_n are a description of the digits of s_{n-1}.

Theorem (Conway, 1987)

Let s_n denote the nth term in the Look-and-Say sequence. Then,

$$\lim_{n \to \infty} \frac{|s_{n+1}|}{|s_n|} = 1.3035\ldots,$$

which is an algebraic integer of degree 71.

The Binary Look-and-Say Sequence is a sequence of binary strings

$$1, 11, 101, 111011, 101110101, 111010110111011, \ldots,$$

in which the bits of s_n are a description of the digits of s_{n-1}.

Theorem (Johnston, 2010 (blog post))

Let s_n denote the nth term in the Binary Look-and-Say sequence. Then,

$$\lim_{n\to\infty} \frac{|s_{n+1}|}{|s_n|} = 1.4655\ldots,$$

which is an algebraic integer of degree 3.

The Look-Knave Sequence is a sequence of binary strings with $s_1 = 1$ in which the bits of s_n are a description of the digits of the bitwise complement of s_{n-1}.

$$s_1 = 1 \qquad \overline{s_1} = 0$$

$$s_2 = 10 \qquad \overline{s_2} = 01$$

$$s_3 = 1011 \qquad \vdots$$

$$\vdots$$

Entries of the Look-Knave sequence

1
10
1011
1011100
1011110101
1011100011101110
10111101111101111011
1011100011101011100011100
1011110111110111011110111110101
101110001110101111011100011101011101110

Conjecture

Let s_n denote the nth term in the Look-Knave sequence. Then,

$$\lim_{n\to\infty} \frac{|s_{n+1}|}{|s_n|} = 1.12\ldots,$$

which is an algebraic integer.

Entries of the Look-Knave sequence

1
1011
1011110101
10111101111101111011
1011110111110111011110111110101
⋮

10
1011100
1011100011101110
1011100011101011100011100
⋮

Theorem (M, 2020)

Let s_n denote the nth term of the Look-Knave sequence. Then the limits

$$\lim_{n\to\infty} s_{2n} = S_{even}, \qquad \lim_{n\to\infty} s_{2n+1} = S_{odd}$$

exist. Further, S_{even} is the description of $\overline{S_{odd}}$ and vice versa.

$$S_{even} = 1011100011101\ldots$$
$$S_{odd} = 1011110111110\ldots$$

Lemma (M, 2020)

Let s_n denote the nth term of the Look-Knave sequence. Then s_n is comprised entirely of the substrings s listed below.

s	$k(s)$	s	$k(s)$
0	10	00011	111100
00	101	0111	11110
000	111	00111	101110
1	10	000111	111110
01	1110	01111	111000
001	10110	001111	1011000
0001	11110	0001111	1111000
011	11100	011111	111010
0011	101100	0011111	1011010

We extend the Knave map to the set of infinite binary strings $2^{\mathbb{N}}$

$$k : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$$

by declaring that

$$k(000\ldots) = 111\ldots$$
$$k(111\ldots) = 000\ldots.$$

Theorem (M, 2020)

The only fixed points of k^2 in $2^{\mathbb{N}}$ are S_{even} and S_{odd}, which are attracting, and `000`... *and* `111`..., *which are repelling.*

Corollary

The infinite binary strings S_{even} and S_{odd} are the unique pair such that S_{even} is the description of $\overline{S_{odd}}$ and vice versa.

Thank you!